Almost Local Field Theories of the S Matrix

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(Received 30 July 1964)

The axioms of local quantum field theory are replaced by the following rather weaker demands: (1) the field $A(x)$, when integrated with a test function, is an unbounded operator in Hilbert space; (2) the field transforms covariantly under a representation $U(a,\Lambda)$ of the Poincaré group which is unitarily equivalent to the representation $U_0(a,\Lambda)$ in a theory of free particles; (3) the field is almost local in the sense of Haag. It is argued that any field satisfying these axioms has a well-defined S matrix which is unitary below the threshold for production processes. The condition that a field is "almost local" is expressed in terms of the Wightman functions of the theory. These conditions can be written down and solved for successively higher Wightman functions for successively larger regions in momentum space; at each stage the problem is well defined with no problems of convergence. A possible connection with Lagrangian theory is given. To illustrate the method the 4-point function is examined in detail below the threshold for production processes. It is found that a necessary condition for "almost locality" is that the scattering amplitude satisfy unitarity in the elastic region. Sufficient conditions in the elastic region are also given. A similar analysis can be carried through for two-channel elastic scattering, and again unitarity appears as a statement of "almost locality."

I. INTRODUCTION

THE difficulty in constructing a local field theory obeying Wightman's axioms[1,2] is due to the fact that any model must, of necessity, be very complicated; it would include all radiative corrections, production processes, etc., and these cannot be accounted for in any simple model. It would seem desirable to allow a slight generalization with a view to getting a workable theory which, while less realistic than a local theory, at least furnishes a satisfactory description of particle scattering and production. If such a field theory is to be useful it must have a well-defined physical interpretation in terms of asymptotic scattering states, at least in a limited energy range. Moreover, it must be possible to modify the theory at any stage so that the physical interpretation can be extended to higher energies. The approach given in this paper has been developed with these points in mind.

If a local field theory has a unique vacuum and nondegenerate one-particle state separated from the continuum, then there is a particle interpretation and an S matrix, given by the Haag-Ruelle collision theory,[3–6] and we recapitulate the main ideas in the next section. This collision theory also works for a class of theories larger than that of local theories, namely, the "almost local" theories of Haag.[3,4,7] It is natural to take almost locality as the appropriate generalization of locality since this generalization does not destroy the possibility of a physical interpretation.

A local or almost local theory is called[8] "asymptotically complete" if the ingoing states (including bound particles) span the Hilbert space. Asymptotic completeness implies that the S matrix is unitary, and this is a very complicated condition. Asymptotic completeness has another immediate consequence, which can be stated simply: the relativistic transformation law $U(a,\Lambda)$ of an asymptotically complete theory is unitary equivalent to that of the theory of the free fields describing the same particles (including bound states). Two theories with the same (or unitarily equivalent up to nonmeasurable phases) transformation laws $U(a,\Lambda)$ are called "relativistically equivalent," a phrase introduced by Wightman.[9] A theory relativistically equivalent to the free fields describing the same particles might not be asymptotically complete, even if it has a collision theory; however, it must have a unique vacuum and a nondegenerate one-particle state, and so if the fields are almost local we can construct the S matrix.

It is a very useful fact that the S matrix of an almost local field theory relativistically equivalent to the free field satisfies unitarity in the elastic region. To see this, note that below the threshold for production processes the states transform according to the representation $[m,s]\otimes[m,s]$ where $[m,s]$ stands for the representation of the Poincaré group with mass m and spin s. In the reduction of this tensor product into irreducible representations[9,10] each irreducible representation $[M,l]$ enters with finite multiplicity, i.e., the dimension of the invariant differential subspace labeled by $[M,l]$ is finite. Now, states with any angular momentum l and any center-of-mass energy $M\geqslant 2m$ can be obtained with the desired multiplicity from the incoming two-particle states; therefore the incoming states span H up to the

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[9] A. S. Wightman, in *Relations de Dispersion et Particules Elêmentaires* (John Wiley & Sons, Inc., New York, 1960).
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onset of inelasticity. Similarly the out-states span H and the elastic S matrix satisfies unitarity.

In view of these remarks, the following set of axioms seems to be the suitable generalization of those of an asymptotically complete local quantum field theory (specialized as usual to the case of a scalar field).

(1) For each test function,[2,11] $f(x)\epsilon S(\mathbf{R}^4)$, $A(f) = \int A(x)f(x)d^4x$ is an (unbounded) operator in a Hilbert space H, defined on vectors in $D\subset H$, and $A(f)D\subset D$.

(2) $A(x)$ transforms as

$$U(a,\Lambda)A(x)U(a,\Lambda)^{-1}=A(\Lambda x+a),$$

where $U(a,\Lambda)$ is unitary equivalent to the representation in a free-field theory of mass m, and $U(a,\Lambda)D\subset D$.

(3) $A(x)$ is an almost local field.

If there are several particles we modify the axioms suitably by introducing several fields and the corresponding free fields in axiom (2). An almost local field is one for which the many-body correlation functions converge to zero faster than any inverse power R^{-n} when the space-time points separate in any way into clusters in a space-like direction, and R is the distance between the clusters. This notion, made precise in the next section, is the relativistic analog of the requirement that the potential shall be of short range. It is called the cluster-decomposition property, and can be proved[12] to lead to the space-like asymptotic property of the S-matrix postulated on physical grounds by Crichton and Wichman.[13] When expressed in terms of the vacuum expectation values, the cluster decomposition property can be classified into subconditions: first according to what order of Wightman function is involved, e.g., 4-point function, 5-point function, etc.; and secondly according to the maximum energy in the test function, assuming it has compact support in momentum space. For example, in this paper we shall consider the four-point function up to the inelastic threshold. A field theory with this limited amount of almost locality has the property that its two-particle states, created at time t, converge strongly as $t\rightarrow\pm\infty$ to in- and out-states. The asymptotic interpretation of the field theory can be extended progressively to more processes as the program is pushed to higher order W functions and higher energies.

In Sec. II the Haag-Ruelle collision theory is reviewed; almost local fields are defined, and it is shown how to define the S matrix for any almost local field. In Sec. III it is explained that any field relativistically equivalent to a free field can be expanded in a series of Wick ordered products in the free field, namely, the Haag expansion. It is shown how the hypothesis that the Hamiltonian is a given functional of the interacting field may be utilized without recourse to the interaction picture. In Secs. IV and V we examine simple consequences of the hypothesis of almost locality; we find that one consequence is that certain functions T related to the four-point functions must satisfy equations identical with the "unitarity" equations; in that case the T's turn out to be the S-matrix elements.

In conclusion, the "almost local program" is compared with the rival S matrix and relativistic potential theories.

II. HAAG-RUELLE COLLISION THEORY

For the sake of completeness we recall the Haag-Ruelle scattering theory in the form it will be used.

A quantized field $A(x)$ may be more singular than a function of x; for mathematical convenience, therefore, Haag introduced[14] the smeared fields

$$A_f(x)=U(x)\int A(x')f(x')d^4x'\;U(x)^{-1}, \qquad (1)$$

where f is a test function $\epsilon S(\mathbf{R}^4)$. The smeared fields have no simple transformation law under the Lorentz group, but do transform covariantly under the translation group, i.e.,

$$U(a)A_f(x)U(a)^{-1}=A_f(x+a) \qquad (2)$$

so that the vacuum expectation values of the smeared fields depend only on the differences of the space-time variables that enter. Next consider the "truncated" vacuum expectation values of $A_f(x)$, defined recursively in terms of the W functions by (W_T denotes truncation)

$$\begin{aligned}\langle A_{f_1}(x_1)\cdots A_{f_n}(x_n)\rangle_0 &= \langle A_{f_1}(x_1)\cdots A_{f_n}(x_n)\rangle_T+\sum_{\text{part}}\langle A_{f i_1}(x_{i_1})\cdots A_{f i_r}(x_{i_r})\rangle_T\\ &\times\langle A_{f j_1}(x_{j_1})\cdots A_{f j_s}(x_{j_s})\rangle_T\cdots\\ &\times\langle A_{f k_1}(x_{k_1})\cdots A_{f k_t}(x_{k_t})\rangle_T, \qquad (3)\end{aligned}$$

where $(i_1,\cdots i_r)(j_1,\cdots j_s)\cdots(k_1,\cdots k_t)$ is a partition of n into parts of r, s, $\cdots$, t elements, and in any part the natural order $i_1<i_2\cdots<i_r$; $j_1<j_2\cdots<j_s$; $\cdots$; $k_1<k_2\cdots<k_t$ is maintained. The sum is over all partitions, and the inductive definition is started by the requirement

$$\langle A_f\rangle_0=\langle A_f\rangle_T \qquad (4)$$

for the one-point function. For $n\geqslant 2$ the truncated part corresponds in perturbation theory to the sum of all connected graphs, and is the product with the contribution from the vacuum intermediate state subtracted out in a symmetrical way in all channels. A field $A(x)$ is

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called almost local if for any test functions $f_1, \cdots f_n \in S$, we have

$$\langle A_{f_1}(t_1,\mathbf{x}_1)\cdots A_{f_n}(t_n,\mathbf{x}_n)\rangle_T \in \mathcal{O}_C'(\mathbf{R}_\xi^{3(n-1)}) \qquad (5)$$

that is, for fixed $t_1, \cdots t_n$ it is a distribution of rapid decrease at ∞ in the variables $\xi_j = \mathbf{x}_j - \mathbf{x}_{j+1}$, $j=1, \cdots n-1$. The physical meaning of this condition has been well given by Haag.[3,4,15] The many-body correlation function falls off rapidly at large distances if the interaction between the particles is of short range. A very important result[5,6] of local field theory is that condition (5) holds if there are no mass-zero particles in the theory.

The physical interpretation of the operator $A_f(x) = \int A(x+y)f(y)d^4y$ may be given in the following way. Construct, at time t, the operator

$$B_\alpha^*(t,f) = \int_{x^0=t} \left\{ A_f(x)\frac{\partial}{\partial x_0} f_\alpha(x) - f_\alpha(x)\frac{\partial}{\partial x_0} A_f(x) \right\} d^3x, \qquad (6)$$

where $f_\alpha(x)$ is a solution of the Klein-Gordon equation with compact support in momentum space. The operator $B_\alpha^*(t,f)$ represents the operation of causing a disturbance, at time t, with wave function related to f and f_α, and having the quantum numbers of the field A. If the field is almost local this interpretation is justified by the Haag-Ruelle theorem: If $B_\alpha^*(t,f)\Psi_0$ is a one-particle state, then

$$\Psi_{\alpha_1\cdots\alpha_n}(t) = B_{\alpha_1}^*(t,f)\cdots \times B_{\alpha_n}^*(t,f)\Psi_0 \to \Psi_{\alpha_1,\cdots\alpha_n}{}^{\mathrm{in,out}} \qquad (7)$$

as $t\to\pm\infty$, where $\Psi_{\alpha_1\cdots\alpha_n}{}^{\mathrm{in,out}}$ are interpreted as ingoing and outgoing n-particle states, and the convergence is in the sense of strong convergence in Hilbert space.

In the definition of the in and out states, only the value of the test function f on the mass shell is relevant; it follows that we can obtain the scattering matrix of a theory by using test functions $\tilde{f}(p)$ which are zero outside a small neighborhood Δ of the mass hyperboloid $p^2=m^2$. Naturally, many different fields give rise to the same S matrix; for example this will be the case if $A_1(f)=A_2(f)$ whenever $\tilde{f}(p)=0$ outside Δ. More generally a creation operator $B^*(t)$ leads to the same S matrix as a creation operator $\hat{B}^*(t)$ if

$$\lim_{t\to\pm\infty} \|[B^*(t)-\hat{B}^*(t)]\Psi_{\alpha_1\cdots\alpha_j}(t)\| = 0. \qquad (8)$$

If $B^*(t)$ and $\hat{B}^*(t)$ are formed from almost local fields $A_f, \hat{A}_f$ as in Eq. (6), then sufficient for (8) is that A and $\hat{A}$ be "almost local with respect to each other"[7] and that the two one-particle states $B^*(t)\Psi_0$ and $\hat{B}^*(t)\Psi_0$ are equal. The method of Ruelle[5,6,16] shows that (8) holds provided that the norm of the state

$$[A_f(x_1)-\hat{A}_f(x_1)]A_f(x_2)\cdots A_f(x_n)\Psi_0$$

is rapidly decreasing in space-like directions. It is in this form that the result will be used, to avoid theories with $S=1$.

The Fourier transform[11] of a distribution in $\mathcal{O}_C'$ is an infinitely differentiable function which increases no faster than a polynomial at ∞ (i.e., a function in $\mathcal{O}_M$). Thus the criterion Eq. (5) for almost locality is

$$\tilde{W}_T(t_1,\mathbf{p}_1;t_2,\mathbf{p}_2;\cdots t_n,\mathbf{p}_n) \in \mathcal{O}_M(\mathbf{p}_1,\cdots\mathbf{p}_{n-1}), \qquad (9)$$

where

$$\int \exp i\sum_1^n \mathbf{p}_i\cdot\mathbf{x}_i \langle A_{f_1}(t_1,\mathbf{x}_1)\cdots A_{f_n}(t_n,\mathbf{x}_n)\rangle_T d^3x_1\cdots d^3x_n = 2(2\pi)^{12}\delta(\sum \mathbf{p})\tilde{W}_T(t_1,\mathbf{p}_1;\cdots t_n,\mathbf{p}_n). \qquad (10)$$

III. FIELDS RELATIVISTICALLY EQUIVALENT TO FREE FIELDS

The representation of the Poincaré group for a theory of a free field of mass m can be reduced to the direct integral of the irreducible representations[17] labeled $[M,l]$. Thus the Hilbert space H of the theory can be written as the direct sum[9,18]

$$H = H_0 \oplus H_1 \oplus H_2 \oplus H_3. \qquad (11)$$

Here H_0 is a one-dimensional subspace corresponding to the identity representation, and is the vacuum state. H_1 consists of one-particle states, and H_2 is the direct integral of invariant differential subspaces labeled by $[M,l]$, and contains the two-particle states with total energy below threshold for three-particle states, namely,

$$H_2 = \int_{2m}^{3m} dM \sum_{l=0}^{\infty} H_{l,M}{}^{(2)}, \qquad (12)$$

where $H_{l,M}{}^{(2)}$ are finite-dimensional. Finally

$$H_3 = \int_{3m}^{\infty} dM \sum_{l=0}^{\infty} H_{l,M}{}^{(3)}, \qquad (13)$$

where each $H_{l,M}{}^{(3)}$ is infinite-dimensional. The action of $U(a,\Lambda)$ on each of these irreducible invariant spaces is known, and therefore the transformation law of any operator in H under $U(a,\Lambda)$ can be determined. For a field $A(x)$ to be relativistically equivalent to the free field we must define the operators $A(f)$ on H so that they transform covariantly under $U(a,\Lambda)$.

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[16] R. F. Streater, Lectures at University of Geneva (unpublished).

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[18] A. S. Wightman, Proceedings of the Midwest Conference in Theoretical Physics, Purdue University, Lafayette, Indiana, 1960 (unpublished).

So far we have considered H and $U(a,\Lambda)$ as an abstract Hilbert space and a representation. We may *realize* the space (to get a concrete space) in any way we like, and there is no harm or loss in generality if we take it as Fock space; this means that the states are given as products of ordinary creation operators $a^\dagger(f)$ on the vacuum state, and the action of $U(a,\Lambda)$ on a given state can be defined in the usual way using the transformation law of the free field $A^0(x)$ corresponding to these creation operators. Our interacting field $A(x)$ is going to be an operator in this Hilbert space, when integrated with a test function. As is well known, every operator must have an expansion in Wick-ordered products of the free field. In order for $A(x)$ to be Lorentz covariant it must have the following Haag[14] expansion (written for simplicity in momentum space, with $\tilde{A}^0(p)=(2p^0)^{1/2}\times\delta(p^2-m^2)[a^\dagger(p)\Theta(p^0)+a(-p)\Theta(-p^0)]$):

$$\begin{aligned}\tilde{A}(p)=\tilde{A}^0(p)+\sum_{m,n=1}^{\infty}\int F_{m,n}(\mathbf{p}_1,\cdots\mathbf{p}_m;\mathbf{q}_1,\cdots\mathbf{q}_n)\\ \times\frac{d^3p_1}{(2\omega_{\mathbf{p}_1})^{1/2}}\cdots\frac{d^3q_n}{(2\omega_{\mathbf{q}_n})^{1/2}}a^\dagger(\mathbf{p}_1)\cdots a^\dagger(\mathbf{p}_m)a(\mathbf{q}_1)\cdots\\ \times a(\mathbf{q}_n)\delta^3(\mathbf{p}_1+\cdots+\mathbf{p}_m-\mathbf{q}_1-\cdots-\mathbf{q}_n-\mathbf{p})\\ \times\delta(\omega_{\mathbf{p}_1}+\cdots+\omega_{\mathbf{p}_m}-\omega_{\mathbf{q}_1}-\cdots-\omega_{\mathbf{q}_m}-p^0),\end{aligned}\quad(14)$$

where the $F_{m,n}$ are Lorentz-invariant functions of $m+n$ 4-vectors defined on the mass shell, and $\omega_{\mathbf{p}}=(\mathbf{p}^2+m^2)^{1/2}$. The permissible class of functions $F_{m,n}$ depends on the class of test functions for which $\tilde{A}(p)$ is to be defined, and the permissible sequences $\{F_{m,n}\}$ depend on the domain of definition we require $A(f)$ to have. We shall not attempt to be more precise, since we shall consider models in which only a finite number of terms enter. The $F_{m,n}$ can in principle be distributions, and we shall find that in fact they must have some sort of "singularity" in order for S to be different from 1.

We do not *a priori* identify the field $A^0(x)$ with either the in field or the out field. If we take $A^0(x)$ to be the in field (assuming it exists), then our theory is asymptotically complete, at least for $t\to-\infty$. In a local theory the out field is then also complete, by the *PCT* theorem.[20,2] In that case the functions $F_{m,n}$ are related to the retarded functions of the field.[21] We have assumed that the interacting field $A(x)$ starts out in the expansion with $A^0(x)$. This involves no loss in generality since we assume that $A(x)$ can create the one-particle state with one application to the vacuum. This is by far the simplest assumption to make. If we integrate $A(x)$ and $A^0(x)$ with a test function g with support near the mass shell in momentum space, then the two-point functions of A and A^0 coincide. This makes it very easy to separate out the vacuum singularities of the Wightman functions in all channels, and enables us to calculate the truncated functions quite easily. This is the main reason why the use of Fock's realization of the Hilbert space and $U(a,\Lambda)$ is so convenient.

We can define the energy operator P^0 as the generator of the time-translation group, and the well-known result is

$$P^0=\int d^3k\omega_k a^\dagger(\mathbf{k})a(\mathbf{k}).\quad(15)$$

It is usually contended that the Hamiltonian defines the dynamics and this appears not to be the case here. However, P^0 cannot be said to be the Hamiltonian unless it is written as a functional of the observables of the theory (remember we are in the Heisenberg picture of quantum mechanics) which in this case could be the smeared field $A(f)$. We divide P^0 up into a "free" part $H_0(t)$ and an "interaction" part $H_1(t)$, both time dependent in general, where we have, perhaps arbitrarily, taken

$$H_0(t)=\int_{x^0=t}T^{00}(x)d^3x,\quad(16)$$

where

$$T^{\mu\nu}(x)=:\frac{\partial A(x)}{\partial x_\mu}\frac{\partial A(x)}{\partial x_\nu}:+\tfrac{1}{2}g_{\mu\nu}\left[m^2:A(x)^2:-:\left(\frac{\partial A(x)}{\partial x^\lambda}\right)^2:\right]\quad(17)$$

is the usual free-field functional. The dots mean that the product must be Wick ordered in the free field after the expansion (14) has been inserted. Under certain conditions on the $F_{m,n}$ functions, the products in (16) and (17) can be made nonsingular. We may obtain dynamics by writing $H_1(t)$ as some well-defined functional of the field $A(x)$, and then requiring that

$$\int d^3k\,\omega_k a^\dagger(k)a(k)=H_0(t)+H_1(t)\quad(18)$$

hold as an operator identity. This will give equations for the expansion functions $F_{m,n}$ in terms of "potentials" introduced in the definition of $H_1(t)$. We do not know that the interacting field satisfies canonical commutation relations and we are anyway not allowed to use the interaction (=Dirac) picture, because an interacting field cannot be unitary equivalent to a free field even if the unitary operator depends on time. Our method [use of Eq. (18)] avoids the interaction picture but is not entirely satisfactory since it is not clear what conditions on $H_1(t)$ will ensure that the resulting field $A(x)$ is almost local. Rather than pursue this problem we simply postulate that $A(x)$ is almost local, thus obtaining directly the desired conditions on the functions $F_{m,n}$.

[19] S. S. Schweber, *Introduction to the Theory of Quantized Fields* (Harper & Row, New York, 1961).
[20] R. Jost, Helv. Phys. Acta **30**, 409 (1957).
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IV. THE ELASTIC SCATTERING AMPLITUDE

Consider a field $A(x)$ having the expansion (14), and choose test functions g_1 and g_2, zero outside sets Δ_1 and Δ_2 in momentum space containing the mass shell, such that the largest momentum component of the state

$$\Psi = A(g_1)A(g_2)\Psi_0$$

is less than the threshold for three particles. Since $A(g_2)\Psi_0$ is in H_1 (the space of one-particle states) the only terms in the expansion of $A(g_1)$ that contribute to Ψ have not more than one annihilation operator in them. Since the state Ψ lies in H_2 the only surviving terms have less than three creation operators. Therefore, when calculating Ψ we can assume that $A(x)$ is given by

$$\tilde{A}(p) = \frac{1}{(2\pi)^2}\int e^{-ip\cdot x}A(x)$$

and

$$\tilde{A}(p) = \tilde{A}^0(p) + \int F(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)\delta^3(\mathbf{p}-\mathbf{k}_1-\mathbf{k}_2+\mathbf{k}_3)\delta(p^0-\omega_1-\omega_2+\omega_3)d^3k_1d^3k_2d^3k_3(8\omega_1\omega_2\omega_3)^{-1/2}a^\dagger(\mathbf{k}_1)a^\dagger(\mathbf{k}_2)a(\mathbf{k}_3), \quad (19)$$

where $\omega_i = (p_i^2+m^2)^{1/2}$. Without loss in generality we can assume F is symmetric in the first two vectors.

We now impose the almost local condition (9) on the four-point truncated function, in this case given by

$$\begin{aligned}\langle A_{g_1}{}^*(t_1,\mathbf{x}_1)A_{g_2}{}^*(t_2,\mathbf{x}_2)A_{g_3}(t_3,\mathbf{x}_3)A_{g_4}(t_4,\mathbf{x}_4)\rangle_T \\ = \langle A_{g_1}{}^*(t_1,\mathbf{x}_1)A_{g_2}{}^*(t_2,\mathbf{x}_2)A_{g_3}(t_3,\mathbf{x}_3)A_{g_4}(t_4,\mathbf{x}_4)\rangle_0 - \langle A_{g_1}{}^*(t_1,\mathbf{x}_1)A_{g_3}(t_3,\mathbf{x}_3)\rangle_0 \\ \times\langle A_{g_2}{}^*(t_2,\mathbf{x}_2)A_{g_4}(t_4,\mathbf{x}_4)\rangle_0 - \langle A_{g_1}{}^*(t_1,\mathbf{x}_1)A_{g_4}(t_4,\mathbf{x}_4)\rangle_0\langle A_{g_2}{}^*(t_2,\mathbf{x}_2)A_{g_3}(t_3,\mathbf{x}_3)\rangle_0, \quad (20)\end{aligned}$$

where we have used Eq. (3) and the fact that $\langle A^*A^*\rangle_0 = 0$. This is a straightforward calculation, and gives for W_T defined in Eq. (10) (using $[a(p),\ a^\dagger(p)] = \delta^3(p-p')$ and $A(g) = \int\tilde{A}(p)g(p)d^4p$):

$$W_T(t_1,\mathbf{p}_1;\ t_2,\mathbf{p}_2;\ t_3,\mathbf{p}_3;\ t_4,\mathbf{p}_4) = W_1+W_2+W_3, \quad (21)$$

with

$$\begin{aligned}W_1 = \bar{g}_1(\omega_1,-\mathbf{p}_1)\bar{g}_2(-\omega_1+\omega_3+\omega_4,-\mathbf{p}_2)g_3(\omega_3,\mathbf{p}_3)g_4(\omega_4,\mathbf{p}_4) \\ \times\exp\{i[-\omega_1t_1-(-\omega_1+\omega_3+\omega_4)t_2+\omega_3t_3+\omega_4t_4]\}\bar{F}(\mathbf{p}_3,\ \mathbf{p}_4,-\mathbf{p}_1)2^{-3}(\omega_1\omega_3\omega_4)^{-1}, \quad (22)\end{aligned}$$

$$\begin{aligned}W_2 = \bar{g}_1(\omega_1,-\mathbf{p}_1)\bar{g}_2(\omega_2,-\mathbf{p}_2)g_3(\omega_1+\omega_2-\omega_4,\ \mathbf{p}_3)g_4(\omega_4,\mathbf{p}_4) \\ \times\exp i[-\omega_1t_1-\omega_2t_2+(\omega_1+\omega_2-\omega_4)t_3+\omega_4t_4]F(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_4)2^{-3}(\omega_1\omega_2\omega_4)^{-1}, \quad (23)\end{aligned}$$

$$\begin{aligned}W_3 = \bar{g}_1(\omega_1,-\mathbf{p}_1)g_4(\omega_4,\mathbf{p}_4)\exp i[-\omega_1t_1+\omega_4t_4]\int\frac{d^3k}{2\omega_k}\exp i[-t_2(\epsilon-\omega_1)+t_3(\epsilon-\omega_4)]\bar{g}_2(\epsilon-\omega_1,-\mathbf{p}_2)g(\epsilon-\omega_4,\ \mathbf{p}_3) \\ \times\bar{F}(\mathbf{k},\ \mathbf{p}_3+\mathbf{p}_4-\mathbf{k},-\mathbf{p}_1)F(\mathbf{k},\ \mathbf{p}_3+\mathbf{p}_4-\mathbf{k},\ \mathbf{p}_4)2^{-3}[\omega_1\omega_4(\epsilon-\omega_k)]^{-1}, \quad (24)\end{aligned}$$

where

$$\epsilon = \omega_k + [(\mathbf{p}_1+\mathbf{p}_2+\mathbf{k})^2+m^2]^{1/2} \quad \text{and} \quad \mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3+\mathbf{p}_4 = 0.$$

The condition that the two-particle states should converge to in and out states is that $W_1+W_2+W_3$ be an infinitely differentiable function of the 3 independent momenta $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$ for all times t_1, t_2, t_3, and t_4.

We can immediately write down a sufficient condition, namely, that F itself is infinitely differentiable. Such a choice, however, leads to $S=1$. To see this, note that the norm of the state

$$\Phi = [A_{g_1}(x_1) - A_{g_1}{}^0(x_1)]A_{g_2}{}^0(x_2)\Psi_0$$

can be expressed in terms of W_3, so that if W_3 is C^∞, then this norm is rapidly decreasing in space-like directions. It follows from the Haag-Ruelle argument that if

$$B(x_1) = A_{g_1}(x_1) - A_{g_1}{}^0(x_1)$$

and

$$B_{f_\alpha}{}^*(t) = \int_{x^0=t} B(x)\frac{\overleftrightarrow{\partial}}{\partial x^0}f_\alpha(x)d^3x$$

then $B_{f_\alpha}{}^*(t)A_{g_2}{}^0(x_2)\Psi_0 \to 0$ as $t\to\pm\infty$, so that A and A^0 have the same asymptotic states, and $S=1$ for the field A.

To get scattering we must ensure that W_3 is not C^∞, but that $W_1+W_2+W_3$ is, i.e., W_1+W_2 must have a "singularity" which cancels one in W_3. Furthermore, since only the field in the neighborhood of the mass shell $p^2=m^2$ is relevant for the S matrix, it is clear that the F's must have some sort of singularity at $p^2=m^2$. In a local field theory with $A^0(x) = A^{\text{in}}(x)$ the F's have the retarded singularity $1/[(p^0+i\epsilon)^2-\mathbf{p}^2-m^2]$, where $\mathbf{p}$ is the momentum of the field; it would therefore be useful to introduce new functions

$$\begin{aligned}T_{m,n}(\mathbf{p}_1,\cdots\mathbf{p}_m;\ \mathbf{q}_1\cdots\mathbf{q}_n) \\ = (p^2-m^2)F_{m,n}(\mathbf{p}_1,\cdots\mathbf{p}_m;\ \mathbf{q}_1,\cdots\mathbf{q}_n), \quad (25)\end{aligned}$$

where

$$\mathbf{p} = \sum_{i=1}^{m}\mathbf{p}_i - \sum_{j=1}^{n}\mathbf{q}_j; \quad p^0 = \sum_{i=1}^{m}\omega_{\mathbf{p}_i} - \sum_{j=1}^{n}\omega_{\mathbf{q}_j}. \quad (26)$$

We will choose the retarded singularity for F; other choices such as advanced or principal value singularities lead to similar analyses. With this choice the singularity in W_1 is of the form

$$\frac{1}{(\omega_3+\omega_4-\omega_1-i\epsilon)^2-\omega_2^2} = \frac{1}{\omega_3+\omega_4-\omega_1+\omega_2}[\omega_3+\omega_4-\omega_1-\omega_2-i\epsilon]^{-1} \quad (27)$$

and the singularity in W_2 is of the form

$$-[\omega_3+\omega_4-\omega_1-\omega_2-i\epsilon]^{-1}[\omega_1+\omega_2-\omega_4+\omega_3]^{-1}. \quad (28)$$

The vectors $\mathbf{p}_i$ are real and satisfy $\mathbf{p}_1+\mathbf{p}_2=\mathbf{p}_3+\mathbf{p}_4$, and so the only possible singularity is where $\omega_1+\omega_2=\omega_3+\omega_4$, which may be called the "energy shell." We notice that all other factors in W_1 and W_2 are the same on the energy shell. Using the mean value theorem we can replace all these factors by their values on the energy shell, provided we add a term proportional to $(\omega_3+\omega_4-\omega_1-\omega_2)$, which cancels with the denominator and eliminates this singularity.

With this remark, then, W_1+W_2 has a singularity of the form

$$\bar{g}_1\bar{g}_2g_3g_4 \exp i[-\omega_1t_1-\omega_2t_2+\omega_3t_3+\omega_4t_4]2^{-4}(\omega_1\omega_2\omega_3\omega_4)^{-1} \times(\bar{T}-T)[\omega_3+\omega_4-\omega_1-\omega_2-i\epsilon]^{-1}, \quad (29)$$

where all the functions are evaluated on the energy shell. This term will have to cancel a nondifferentiable part of W_3.

To evaluate W_3 we work in a particular Lorentz frame. This is permissible since T is Lorentz invariant (see Appendix) and defined in a region of momentum space which is connected under the real proper Lorentz group. Choose

$$\begin{aligned} p_1&=(s^{1/2}/2,\ (s/4-m^2)^{1/2},\ 0,\ 0), \quad p_2=(s^{1/2}/2,\ -(s/4-m^2)^{1/2},\ 0,\ 0), \\ p_3&=(-s^{1/2}/2,\ (s/4-m^2)^{1/2}\cos\theta_c,\ (s/4-m^2)^{1/2}\sin\theta_c,\ 0), \\ p_4&=(-s^{1/2}/2,\ -(s/4-m^2)^{1/2}\cos\theta_c,\ -(s/4-m^2)^{1/2}\sin\theta_c,\ 0), \\ k&=((\mathbf{k}^2+m^2)^{1/2},\ k\cos\theta,\ k\sin\theta\cos\varphi,\ k\sin\theta\sin\varphi). \end{aligned}$$

We now change the variables of integration in Eq. (24) from k, θ, φ to ω_k, θ, φ; then $d^3k=\omega_k(\omega_k^2-m^2)^{1/2}d\omega_k d\Omega$ and the ω_k integration goes from m to $+\infty$.

The functions $\bar{F}(\mathbf{k},\mathbf{p}_1+\mathbf{p}_2-\mathbf{k},\mathbf{p}_1)$ and $F(\mathbf{k},\mathbf{p}_1+\mathbf{p}_2-\mathbf{k},-\mathbf{p}_4)$ have poles, respectively, at

$$\{\omega_k+[(\mathbf{p}_1+\mathbf{p}_2-\mathbf{k})^2+m^2]^{1/2}-\omega_1\}^2=\mathbf{p}_2^2+m^2$$

and at

$$\{\omega_k+[(\mathbf{p}_1+\mathbf{p}_2-\mathbf{k})^2+m^2]^{1/2}-\omega_4\}^2=\mathbf{p}_3^2+m^2.$$

In the particular coordinate system the poles become $[(2\omega_k-\omega_1-i\epsilon)^2-\omega_2^2]^{-1}$ and $[(2\omega_k-\omega_4+i\epsilon)^2-\omega_3^2]^{-1}$ and these fall on the line of integration of ω_k where $\omega_k=\frac{1}{2}(\omega_1+\omega_2+i\epsilon)$ and $\frac{1}{2}(\omega_3+\omega_4-i\epsilon)$. If the integrand were an analytic function then the well-known analysis tells us that the singularities in the external variables $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$ come either from a pinch, i.e., when $\omega_1+\omega_2=\omega_3+\omega_4$, or possibly but not inevitably from an end-point singularity $\omega_1+\omega_2=2m$ or $\omega_3+\omega_4=2m$. If T is not analytic but is infinitely differentiable then a simple discussion of the integrals shows that the same result holds. The singular part of W_3 due to the pinch is

$$\begin{aligned} &\pi i\bar{g}_1\bar{g}_2g_3g_4 \exp i(-\omega_1t_1-\omega_2t_2+\omega_3t_3+\omega_4t_4) \\ &\quad\times\int \bar{T}(\mathbf{k},\mathbf{p}_1+\mathbf{p}_2-\mathbf{k},\mathbf{p}_1)T(\mathbf{k},\mathbf{p}_1+\mathbf{p}_2-\mathbf{k},-\mathbf{p}_4) \\ &\quad\times d\Omega(\omega_k^2-m^2)^{1/2}(16\omega_1\omega_2\omega_3\omega_4)^{-1}(2\omega_k)^{-1} \\ &\quad\quad\times[2\omega_k-\omega_3-\omega_4+i\epsilon]^{-1}, \quad (30) \end{aligned}$$

where k is such that there is a pinch, namely, $\omega_1+\omega_2=\omega_3+\omega_4$ and $2\omega_k=\omega_1+\omega_2$, and all functions are evaluated at this value.

If $\bar{T}T$ can be analytically continued as a function of ω_k, we get the form (30) by deforming the contour over the pole $2\omega_k=\omega_1+\omega_2+i\epsilon$ thus picking up the residue (30). The remaining integral will be C^∞ except possibly for the end-point singularity. In order for the singularity (30) of W_3 to cancel the singularity (29) of W_1+W_2, the following equation must hold:

$$\bar{T}-T=+\pi i\int\bar{T}T\left(\frac{s-4m^2}{4s}\right)^{1/2}d\Omega, \quad (31)$$

where we have used $(\omega_k^2-m^2)^{1/2}/2\omega_k=(s-4m^2/4s)^{1/2}$ in this coordinate system. This is just the usual elastic unitarity relation on the mass shell. Solutions of this equation may be expected to have a branch point of the form $(s-4m^2)^{1/2}$, and this is not differentiable at one point in momentum space, namely $\mathbf{p}_1=\mathbf{p}_2=0$. This branch point might cause a singularity in W_1+W_2 (apart from the pole we have just canceled) where $\omega_1+\omega_2=2m$, and this threshold singularity must cancel with the end-point singularity in W_3, which also occurs when $\omega_1+\omega_2=2m$ or $\omega_3+\omega_4=2m$. A possible solution to this delicate problem is to eliminate both threshold and end-point singularities by choosing the function T to rise very smoothly from zero at threshold, for example, like $e^{-1/(s-4m^2)}$. Such a threshold behavior, together with Eq. (31), are sufficient for the truncated function under consideration to satisfy the almost local condition. Therefore we can proceed with the Haag-Ruelle limit, and calculate the S matrix in the elastic region.

Since the state $\Psi(t)=B_{\alpha_1}{}^*(t,f)B_{\alpha_2}{}^*(t,f)\Psi_0$ converges in norm as $t\rightarrow\pm\infty$ we can determine the in and out states by evaluating the matrix element

$$(\Psi_0,a(p_1)a(p_2)\Psi(t))\text{ which converges at } t=\pm\infty\,.$$

This matrix element is the same as for the free field modified by the addition of the term

$$\left(\Psi_0,a(\mathbf{p}_1)a(\mathbf{p}_2)\int_{x^0=t} d^3x\, f_{\alpha_1}(x)\frac{\overleftrightarrow{\partial}}{\partial x^0}B_1{}^*(x)B_{\alpha_2}{}^*(t,f)\Phi_0\right), \tag{32}$$

where

$$B_1{}^*(x)=U(x)\bigg\{\int T(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)a^\dagger(\mathbf{k}_1)a^\dagger(\mathbf{k}_2)a(\mathbf{k}_3)[(p^0-i\epsilon)^2-\mathbf{p}^2-m^2]^{-1}$$
$$\times\tilde{g}(p)d^4pd^3k_1d^3k_2d^3k_3[8\omega_1\omega_2\omega_3]^{-1/2}\delta^3(\mathbf{p}-\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)\delta(p^0-\omega_1-\omega_2+\omega_3)\bigg\}U(x)^{-1}. \tag{33}$$

The limit of (32) as $t\rightarrow-\infty$ vanishes because of the retarded property of the propagator, using the methods of Ref. 21. Thus the 2 particle in states are the same as those created by the basic field $A^0(x)$. The scattering amplitude $(\Psi^{\text{out}},\Psi^{\text{in}})$ is therefore given by the complex conjugate of the limit of (32) as $t\rightarrow+\infty$, and this is easily proved to be $\bar{T}(\mathbf{p}_1,\mathbf{p}_2,-\mathbf{p}_4)$ evaluated on the energy shell $\omega_1+\omega_2=\omega_3+\omega_4$. It is not surprising that elastic unitarity turns out to be a necessary condition for almost locality, in view of the remarks made in the introduction about theories relativistically equivalent to the free field. It is interesting that the argument breaks down in the production region.

V. TWO-CHANNEL ELASTIC SCATTERING

The method of the previous section can be generalized to any scattering, production or rearrangement collision at finite energies. The next simplest example is when there are two particles in the theory with the same or different quantum numbers. If the particles have masses m_A, m_B with $m_B>m_A$, and if $2m_B<3m_A$ we can discuss the "elastic" region of the 9 processes $A+A$, $A+B$, $B+B\rightarrow A+A$, $A+B$, $B+B$. The representation of the Poincaré group is the same as that for two free fields $A^0(x)$, $B^0(x)$ of masses m_A, m_B. We assume that there are fields $A(x)$, $B(x)$ describing the two particles and we expand them in terms of the free fields, similar to Eq. (14), with $A(x)$ beginning with $A^0(x)$ and $B(x)$ with $B^0(x)$. If $a^\dagger$, $b^\dagger$ are the creation operators of A^0 and B^0 then we need only keep those terms in the expansions of A and B involving the products $a^\dagger a^\dagger a$, $a^\dagger b^\dagger b$, $a^\dagger a^\dagger b$, $a^\dagger b^\dagger a$, $b^\dagger b^\dagger a$, and $b^\dagger b^\dagger b$. There are 9 different truncated functions on which we impose the condition of almost locality. We assume that A has a singularity only at $p^2=m_A{}^2$ and B has one only at $p^2=m_B{}^2$. This involves no loss of generality even if A and B have the same quantum numbers, since we can always make a linear transformation to eliminate the mixing.

If we smear $A(x)$ and $B(x)$ with test functions nonzero only near the respective mass shells, and pursue the analysis leading to Eq. (31) we find that indeed all the usual physical unitarity equations are forced on us in order to cancel the singularities, together with suitable smoothness conditions, as before. By looking only in the physical region (i.e., for real vectors $\mathbf{p}_i$) we can obtain only the physical unitarity condition. For example we learn nothing about the discontinuity of the amplitude $B+B\rightarrow B+B$ below the threshold $2m_B$, and so do not obtain the usual relations involving $A+A$ and $A+B$ intermediate states. Indeed we have no way of defining unphysical amplitudes unless we assume some sort of analyticity; we can always eliminate threshold singularities by drastic smoothness assumptions like $\exp(-(s-4m^2)^{-1})$. In a *local* theory where we have analyticity it is much harder to satisfy the condition of almost locality, and unphysical unitarity is the natural way to ensure that the singularities cancel.

We have assumed that both particles A and B are described by almost local fields. It is conceivable that a more general analysis is possible wherein we assume only that there is one field $A(x)$ which has both singularities at $p^2=m_A{}^2$ and $p^2=m_B{}^2$, or even more generally, that the B particle is created by some polynomial in the field $A(x)$; it might be argued that this would be the case if B were a true bound state, and not elementary. The most general description would be to assume that B is created by an almost local polyfield $B(x_1,\cdots x_m)$; this is defined as a polyfield[22,23] such that there is a test function f such that

$$B(0)=\int B(x_1,\cdots x_m)f(x_1,\cdots x_m)d^4n_1\cdots d^4x_m\Psi_0$$

is a one-particle B state, while the field $B(x)=U(x)B(0)U(x)^{-1}$ satisfies the cluster decomposition property with itself and the other field $A(x)$ in the theory. However, for a local field theory it is likely that the bound states always have a local field to describe them,[24] and so it is improbable that a discussion of polyfields leads to any real generalization.

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VI. CONCLUSIONS

The axioms (1), (2), and (3) for an almost local field have been chosen so that, on the one hand, they are not so restrictive that it is difficult to get a model, and on the other hand they are not so general that a physical interpretation is impossible. There is some hope that a proof of the existence of a model satisfying (1), (2), and (3) can be found; in this paper the almost locality condition is examined only in the elastic region, and it is found to lead to equations similar to "physical" unitarity for which it is easy to find solutions.

In this analysis there is no reason to suppose that the functions involved are analytic, and we would expect that it is much easier to ensure the required smoothness of the functions in momentum space by choosing infinitely differentiable functions with a threshold behavior like $\exp[-1/(s-4m^2)]$. Essential singularities like this are not allowed in so-called "analytic S-matrix theory." Apart from analyticity and crossing symmetry, the problems arising are very similar to S-matrix theory. The extra properties of analyticity and crossing symmetry which can certainly be imposed on our formalism, are the natural consequences of a *local* field. It may be possible to restrict the possible theories by such assumptions, but it is important not to impose them too rigidly on any model field $A(x)$ which has a finite expansion in the free field, since this leads to contradictions.[25]

The axioms (1), (2), and (3) imply that the S matrix is unitary in the elastic region, but do not imply 3-particle unitarity. Even if 3-particle unitarity is not satisfied, any model will be capable of a good description of nature, because it will contain all two-particle final-state interactions in the 3-particle scattering amplitude. It is to be expected that any local theory can be well approximated by simple almost local models.

Compared with relativistic potential scattering,[26] almost local theory is more versatile, in that it allows production of arbitrary numbers of particles in a collision; it is not clear whether production is *implied* by the axioms, as it is for local theory. In the elastic region, potential theory and almost-local field theory ought to be closely connected, since both are motivated by relativity, and the cluster decompostion property. Close comparison, however, is not easy, since field theory does not make use of the dynamical group $U(t)$ and the Schrödinger picture. It is clear that in an almost local field theory which is not local the transformation from the free to the interacting fields is not even canonical, let alone unitary. It may be possible to throw light on the connection between the theories by investigating the Hamiltonian in an almost local theory, and in particular, by studying equations of motion.

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ACKNOWLEDGMENTS

The author is indebted to Professor J. M. Jauch for the hospitality of the Institute for Theoretical Physics, Geneva, where this work was started, and Professor R. G. Sachs, Director of the Summer Institute in Theoretical Physics, where the work was completed. He thanks Dr. O. Steinmann for reading the manuscript.

APPENDIX

We now briefly justify the method used to evaluate W_3 in Eq. (4). Although W_3 is not a Lorentz-invariant function it can be written as $(\sum p=0)$

$$\delta^3(\mathbf{p}_1+\mathbf{p}_2-\mathbf{p}_3-\mathbf{p}_4)W_3(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3;t_1,t_2,t_3,t_4) = \int \hat{W}(p_1,\cdots p_4)\bar{g}_1(\mathbf{p}_1,p_1^0)\cdots g_4(\mathbf{p}_4,p_4^0) \times \exp i[p_1^0t_1+\cdots+p_4^0t_4], \quad \text{(A1)}$$

where $\hat{W}$ is Lorentz invariant. If p is a four-vector, let us denote by Λp^0 and $\Lambda\mathbf{p}$ the time and space components of Λp respectively, where Λ is a Lorentz transformation such that $\Lambda(\mathbf{p}_1+\mathbf{p}_2)=0$.

This Λ is nonunique but its effect on the time components is unique. Then

$$\begin{aligned} W_3 &= \int \hat{W}(\Lambda p_1,\cdots\Lambda p_4)\bar{g}_1(\mathbf{p}_1,p_1^0)\cdots g_4(\mathbf{p}_4,p_4^0) \times \exp i[p_1^0t_1+\cdots+p_4^0t_4]dp_1^0\cdots dp_4^0 \\ &= \int \hat{W}(\Lambda\mathbf{p}_1,\cdots\Lambda\mathbf{p}_4;p_1^0,\cdots p_4^0) \\ &\quad\times \bar{g}_1(\mathbf{p}_1,\Lambda^{-1}p_1^0)\cdots g_4(\mathbf{p}_4,\Lambda^{-1}p_4^0) \times \exp i[p_1^0\Lambda t_1+\cdots+p_4^0\Lambda t_4]. \quad \text{(A2)} \end{aligned}$$

From (A2) we can evaluate the pinch at the new point $\Lambda\mathbf{p}_1, \cdots\Lambda\mathbf{p}_4$ as above, and then transform back with Λ^{-1} to get Eq. (30).